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POSTBUCKLING ANALYSIS OF VISCOELASTIC LAMINATED PLATES USING HIGHER-ORDER THEORY

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Abstract—The postbuckling behavior of viscoelastic composite laminated plates is considered. The viscoelastic behavior of the single ply is characterized by a micro-mechanical theory, in conjunction with the properties of the fibers and viscoelastic resin matrix. Higher-order shear-deformation theory is employed to study the post-buckling phenomenon. The resulting viscoelastic effects are shown, and comparison with lower order theories is presented.

1. INTRODUCTION

The buckling state of a structure is an important measure of the allowable loading. It is, however, well known that a better utilization of the structure is achieved beyond the buckling point, extending its performance into the postbuckling region. As composite materials become extensively used in the aeronautical and aerospace industry, it is essential to extend the postbuckling analysis to laminated composite plates. The postbuckling analysis of perfectly elastic laminated plates was presented recently by Stein (1983, 1985) and Zhang and Matthews (1984), for example. A comprehensive list of references can be found in recent reviews by Leissa (1987) and Kapania and Raciti (1989).

Resin matrix composites (e.g. graphite/epoxy) are well known to exhibit appreciable viscoelastic behavior. This is due to the existence of the polymeric matrix which shows time-dependent effects. A comprehensive review of the viscoelastic behavior and analysis of composite materials was given by Schapery (1974). Since the overall behavior of unidirectional composites can be described by a transversely isotropic material, one needs to provide five independent creep functions for the characterization of the viscoelastic behavior. It seems that the determination of these functions is cumbersome and expensive as it requires an extensive testing program. However, a micro-mechanical approach can readily provide these time-dependent functions. This approach is based on the knowledge of the properties of the fibers and the isotropic matrix. Such a micro-mechanical methodology can be found in a recent review by Aboudi (1989). This micro-mechanical analysis was applied by Yancey and Pindera (1990) for the prediction of the behavior of unidirectional composites consisting of viscoelastic epoxy matrices. It was shown that the micro-mechanical prediction correlated well with various experimental results. In Cederbaum and Aboudi (1989), the buckling load of orthotropic viscoelastic plates was predicted. Similarly, Chandiramani *et al.* (1989) investigated the dynamic stability of viscoelastic orthotropic panels. In both papers, the viscoelastic properties of the composite were determined by utilizing the above micromechanical analysis.

In the present paper, the postbuckling analysis of laminated plates made of viscoelastic resin matrix composites is given. The viscoelastic characterization of the unidirectional layer relies on the micro-mechanical formulation mentioned above in conjunction with the viscoelastic behavior of the matrix. The nonlinear analysis for the postbuckling behavior is formulated within the framework of a higher-order shear-deformation plate theory. Classical, as well as first-order plate theories are obtained as special cases of the present derivation. For perfectly elastic plates, first-order and higher-order plate theories were presented by Ambartsumian (1970), Librescu (1975), Reddy (1984a,b, 1987) and Librescu and Reddy (1987). It was shown in these investigations that the incorporation of transverse shear might be significant for anisotropic plates.

In order to assess the accuracy of the present analysis, we present the postbuckling behavior of perfectly elastic laminated plates composed of boron epoxy, glass/epoxy and graphite/epoxy. For these cases, the results of Chia (1980) are used for comparison. For viscoelastic composite laminated plates, results of the postbuckling behavior are given for a boron/epoxy material system. Deflection versus axial compressive load results are presented for cross-ply laminates with various numbers of layers at several time intervals. The effect of the higher-order formulation is studied by comparisons with the results based on first-order and classical theories. The results clearly exhibit the effect of the viscoelastic matrix on the postbuckling behavior.

2. OVERALL COMPOSITE BEHAVIOR

2.1. *Composite elastic constants*

Let $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ denote a Cartesian coordinate system with $\bar{x}_1$ oriented in the fiber direction of a unidirectional, fiber-reinforced, elastic composite. The constitutive law for the effective transversely isotropic behavior of such a composite is given by a micro-mechanical analysis developed by Aboudi (1987):

$$\bar{\sigma} = \mathbf{E}\bar{\varepsilon}, \tag{1}$$

where $\bar{\sigma}$ and $\bar{\varepsilon}$ are the average stress and strain in the composite, respectively, and $\mathbf{E}$ is the effective stiffness tensor which describes the transversely isotropic behavior. The explicit expressions for the elements of $\mathbf{E}$ in terms of the fiber and matrix properties and the reinforcement volume ratio can be found in Aboudi (1989).

2.2. *Composite viscoelastic representation*

With $\mathbf{E}$ in eqn (1) representing the five independent elastic constants of the equivalent transversely isotropic material which represents the unidirectional composite, it is possible to obtain the five time-dependent functions which characterize the viscoelastic composite whose constituents are viscoelastic materials. Each phase ($\alpha = 1, 2$) is represented by Boltzmann's superposition principle (Christensen, 1982) in tensorial notation as follows,

$$\sigma_{ij}^{(\alpha)}(t) = \int_{-\infty}^{t} c_{ijkl}^{(\alpha)}(t-\tau)\dot{\varepsilon}_{kl}^{(\alpha)}(\tau)\,\mathrm{d}\tau, \tag{2}$$

where $c_{ijkl}^{(\alpha)}(t)$ are the relaxation functions of the phase (throughout this paper, Greek indices have the values 1 and 2 and Latin indices run over 1, 2 and 3). By applying the Laplace transform to eqn (2) we obtain,

$$L[\sigma_{ij}^{(\alpha)}] = sL[c_{ijkl}^{(\alpha)}]\,L[\varepsilon_{kl}^{(\alpha)}], \tag{3}$$

where $L[\cdot]$ denotes the Laplace transform and s is the transform parameter. Thus, the use of $sL[c_{ijkl}^{(\alpha)}]$ in the micro-mechanics analysis (instead of $c_{ijkl}^{(\alpha)}$ in the elastic case), provides $sL[\mathbf{E}(t)]$. It follows that in the transform domain, the stress–strain constitutive law of the composite is given according to (1) and (3) by

$$L[\bar{\sigma}(t)] = sL[\mathbf{E}(t)]\,L[\bar{\varepsilon}(t)]. \tag{4}$$

The inversion of $L[\mathbf{E}(t)]$ back to the time domain provides the relaxation functions $\mathbf{E}(t)$, while the solution of any laminate, composite-structure problem yields the corresponding solution in the transform domain by replacing $\mathbf{E}$ with $sL[\mathbf{E}]$. The inversion to the time domain can be performed by adopting the numerical method of Bellamn *et al.* (1966).

3. ELASTIC POSTBUCKLING BEHAVIOR

3.1. *Kinematics*

The postbuckling analysis of cross-ply symmetric laminates is considered. In the Cartesian coordinate system of the plate x_1, x_2 and x_3 in which $\bar{x}_3 = x_3$, the corresponding displacements are u_1, u_2 and u_3. In the framework of a higher-order shear deformation plate theory, the displacements u_1 and u_2 are expanded as cubic functions of the thickness coordinate x_3 and the transverse deflection is assumed to be constant,

$$u_\alpha(x_1,x_2,x_3) = u_\alpha^0(x_1,x_2)+x_3u_\alpha^1(x_1,x_2)+(x_3)^2u_\alpha^2(x_1,x_2)+(x_3)^3u_\alpha^3(x_1,x_2)$$
$$u_3(x_1,x_2) = u_3^0(x_1,x_2). \tag{5}$$

Here, superscripts 0, 1, 2 and 3 refer to the order of expansion of the displacements. It should be noted that superscipt 0 corresponds to displacements on the midplane, superscript 1 to rotations of normals to the midplane about direction α, and superscripts 2 and 3 correspond to functions to be determined by the conditions on the top and bottom surfaces of a laminate of thickness h,

$$m_\alpha = \sigma_{\alpha 3}x_3|_{-h/2}^{+h/2} \to 0, \tag{6}$$

where $\sigma_{\alpha 3}$ are the transverse shear components of the stress tensor and m_α are the load couple components (measured per unit area of the reference surface) taken to be zero.

By using the partial nonlinear strain–displacement relationship in which the nonlinearity is taken in the 3rd direction only, i.e.

$$e_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}+u_{3,i}u_{3,j}) \tag{7}$$

in which e_{ij} are the strain components of the laminate, we obtain the following expression for the transverse shear strains,

$$e_{\alpha 3} = \tfrac{1}{2}(u_{\alpha,3}+u_{3,\alpha}) \tag{8}$$

since $u_{3,3} = 0$. Substitution of the displacements given by (5), yields

$$e_{\alpha 3} = \tfrac{1}{2}[u_\alpha^1+2x_3u_\alpha^2+3(x_3)^2u_\alpha^3+u_{3,\alpha}^0]. \tag{9}$$

3.2. *Constitutive equations*

From constitutive eqn (1) we obtain the expression of transverse shear stress–strain relations,

$$\sigma_{\alpha 3} = E_{\beta 3\alpha 3}[u_\beta^1+2x_3u_\beta^2+3(x_3)^2u_\beta^3+u_{3,\beta}^0]; \tag{10}$$

employing eqn (10) in conjunction with (6) yields,

$$u_\alpha^3 = -\frac{4}{3h^2}(u_{3,\alpha}^0+u_\alpha^1) \tag{11}$$

and u_α^2 vanishes.

Using the above displacement expansion [eqn (5)] the following expressions for the strain components can be obtained,

$$e_{\alpha\beta} = e_{\alpha\beta}^0+x_3e_{\alpha\beta}^1+(x_3)^3e_{\alpha\beta}^3$$
$$e_{\alpha 3} = e_{\alpha 3}^0+(x_3)^2e_{\alpha 3}^2$$
$$e_{33} = 0. \tag{12}$$

When eqn (12) is written explicitly, products of the in-plane displacements or their derivatives are neglected. The constitutive relations are now expressed as follows,

$$\sigma_{\alpha\beta} = \tilde{E}_{\alpha\beta\xi\eta}e_{\xi\eta}+\delta\frac{E_{\alpha\beta 33}}{E_{3333}}\sigma_{33} \tag{13}$$

in which

$$\tilde{E}_{\alpha\beta\xi\eta} = E_{\alpha\beta\xi\eta}-\frac{E_{\alpha\beta 33}E_{33\xi\eta}}{E_{3333}}.$$

3.3. *Stress and moment resultants*

From the three-dimensional equations of equilibrium,

$$\sigma_{3\alpha,\alpha}+\sigma_{33,3} = 0. \tag{14}$$

Following Mindlin (1951), we integrate eqn (14) over the thickness x_3 to obtain,

$$\sigma_{33} = x_3\sigma^1_{33}+(x_3)^2\sigma^2_{33}+(x_3)^3\sigma^3_{33}, \tag{15}$$

where superscripts 1, 2 and 3 denote the coefficients in the expansion of σ_{33} (as previously used for the expansions of the displacements and the strains).

Using eqn (12) in eqn (13) in conjunction with eqn (11) provides

$$\sigma_{33} = -x_3E_{\alpha 3\beta 3}\left[(u^1_{\beta,\alpha}+u^0_{3,\beta\alpha})\left(1+\frac{4(x_3)^2}{3h^2}\right)+x_3u^0_{3,\alpha}u^0_{3,\beta}\frac{4}{h^2}\right]. \tag{16}$$

By integrating the stresses through the thickness and employing eqns (10), (12) and (13), we obtain

$$N_{\alpha 3} = (u^1_\beta+u^0_{3,\beta})(A_{\alpha 3\beta 3}-B_{\alpha 3\beta 3}) \tag{17}$$

$$N_{\alpha\beta} = (u^0_{\xi,\eta}+u^0_{\eta,\xi}+u^0_{3,\xi}u^0_{3,\eta})H_{\alpha\beta\xi\eta}-u^0_{3,\xi\eta}I_{\alpha\beta\xi\eta}-(u^0_{3,\xi}u^0_{3,\eta})J_{\alpha\beta\xi\eta} \tag{18}$$

$$M_{\alpha\beta} = (u^1_{\xi,\eta}+u^1_{\eta,\xi})(C_{\alpha\beta\xi\eta}-D_{\alpha\beta\xi\eta})-u^0_{3,\xi\eta}2D_{\alpha\beta\xi\eta}-[(u^1_{\xi,\eta}+u^0_{3,\xi\eta})(F_{\alpha\beta\xi\eta}-G_{\alpha\beta\xi\eta})] \tag{19}$$

in which A, B, C, D, F, G, H, I and J are the tensorial rigidities given explicitly in the Appendix.

3.4. *Governing equations*

In the absence of lateral loads, the two-dimensional equilibrium equations are,

$$N_{\alpha\beta,\alpha} = 0 \tag{20}$$

$$M_{\alpha\beta,\beta}-N_{\alpha 3} = 0 \tag{21}$$

$$N_{\alpha\beta}u^0_{3,\alpha\beta}+N_{\alpha 3,\alpha} = 0. \tag{22}$$

Equations (20)–(22) yield five independent relations which govern the elastic field in the laminated plate. The unknown functions are $u^{0,1}_1$, $u^{0,1}_2$ and u^0_3, which are governed by eqns (20)–(22). Substituting eqn (18) into the governing eqn (21), and noting that the rigidities with odd numbers of likewise indices vanish for the case of cross-ply symmetric laminates, we obtain the following explicit equations,

$$u^1_{1,11}[2(C_{1111}-D_{1111})]+u^1_{2,21}[2(C_{1122}-D_{1122})+2(C_{1212}-D_{1212})]$$
$$-u^0_{3,111}(2D_{1111})-u^0_{3,221}(2D_{1122}+4D_{1212})+u^1_{1,22}[2(C_{1212}-D_{1212})]$$
$$+(u^1_1+u^0_{3,1})(A_{1313}-B_{1313})-\{(u^1_{1,11}+u^0_{3,111})(F_{1111}+G_{1111})+u^1_{2,21}(F_{1122}+G_{1122})$$
$$+u^0_{3,221}(F_{1122}+G_{1122})\} = 0, \qquad (23\text{–}24)$$

where eqn (24) is obtained from eqn (23) (as written above) by interchanging indices 1 and 2 throughout the entire equation. Similarly, eqn (22) provides

$$u^0_{3,11}\{[(u^0_{3,1})^2H_{1111}+(u^0_{3,2})^2H_{1122}]-[u^0_{3,11}I_{1111}+u^0_{3,22}I_{1122}]-[(u^0_{3,1})^2J_{1111}+(u^0_{3,2})^2J_{1122}]\}$$
$$+4u^0_{3,12}\{[(u^0_{3,1}u^0_{3,2})H_{1212}-u^0_{3,12}I_{1212}]\}+u^0_{3,22}\{[(u^0_{3,2})^2H_{2222}+(u^0_{3,1})^2H_{2211}]$$
$$-[u^0_{3,22}I_{2222}+u^0_{3,11}I_{2211}]-[(u^0_{3,2})^2J_{2222}+(u^0_{3,1})^2J_{2211}]\}$$
$$+(u^1_{1,1}+u^0_{3,11})(A_{1313}-B_{1313})+(u^1_{2,2}+u^0_{3,22})(A_{2323}-B_{2323}) = 0. \qquad (25)$$

Equations (23)–(25) form a portion of the nonlinear system which governs the stability and postbuckling behavior for cross-ply symmetric laminates, in which u^0_1, u^0_2 terms are not given.

The boundary conditions for the case of simply supported, uniaxially loaded rectangular laminated plates are,

$$u^0_3 = N_{12} = u^1_2 = M_{11} = 0, \quad N_{11} = T_{11}, \quad \text{on } x_1 = 0, a$$
$$u^0_3 = N_{12} = u^1_1 = M_{22} = N_{22} = 0, \quad \text{on } x_2 = 0, b \qquad (26)$$

in which T_{11} is the axial external load and a and b are the plate dimensions in the x_1 and x_2 directions, respectively.

The set of three equations (23), (24) and (25) are linearized, following the adaptation of Newton's technique in which the nonlinear terms are developed into a Taylor series, as presented by Thurston (1965). The resulting linearized system is solved by the Galerkin method in which the orthogonalization product of the weighted residual is integrated exactly using the symbolic manipulation package muMATH. The following are the trial functions which meet the requirement of being comparison functions which fulfill the above boundary conditions in the displacements and are from a complete set,

$$u^0_3 = \sum_{m,n=1}^{N} W_{mn} \sin(\alpha x_1) \sin(\beta x_2)$$
$$u^{0,1}_1 = \sum_{m,n=1}^{N} X^{0,1}_{mn} \cos(\alpha x_1) \sin(\beta x_2)$$
$$u^{0,1}_2 = \sum_{m,n=1}^{N} Y^{0,1}_{mn} \sin(\alpha x_1) \cos(\beta x_2), \qquad (27)$$

in which $\alpha = m\pi/a$, $\beta = n\pi/b$ and N is the number of terms in the series. Expression (27) leads to a set of $5N$ linear algebraic equations in the unknown vectors $\{X^{0,1}_{mn}\}$, $\{Y^{0,1}_{mn}\}$ and $\{W_{mn}\}$. The stress and moment resultant related boundary conditions are imposed by employing the Galerkin method as well. So far, the analysis was carried out for the elastic laminated plates. As mentioned before, the viscoelastic effects of the fiber and matrix phases are incorporated by replacing the elastic **E** tensor by the product of the Laplace parameter s with the Laplace transform of $\mathbf{E}(t)$.

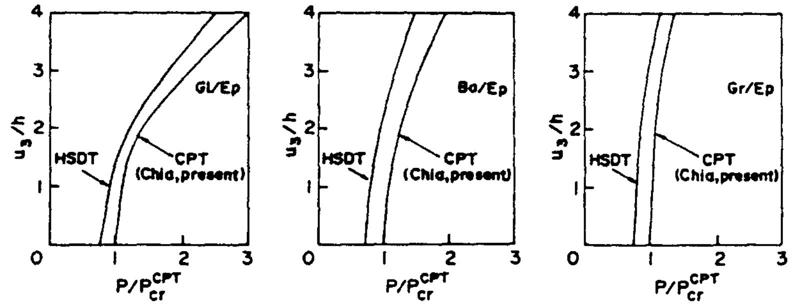

Fig. 1. Central deflection vs postbuckling load of a unidirectional composite elastic plate (square, simply supported), predicted by CPT (Chia, 1980 and present analysis) and HSDT for glass/epoxy, boron/epoxy and graphite/epoxy material systems.

4. APPLICATIONS

A verification of the accuracy of our results was performed by a comparison with the solution given by Chia (1980) for the postbuckling behavior of three types of perfectly elastic, unidirectional composite plates using the classical lamination theory—CPT. The composite plates considered are composed of boron/epoxy, glass/epoxy and the graphite/epoxy material systems. The computed postbuckling curves obtained by the two methods coincide. In Fig. 1, a comparison between the postbuckling behavior of a perfectly elastic, unidirectional square plate (all edges are simply supported) subjected to uniaxial compression is shown. Two types of theories have been used in generating the postbuckling behavior: CPT and higher-order shear deformation theory—HSDT. In this figure, the central deflection u_3 (normalized with respect to the thickness h of the plate) is shown versus the postbuckling load P (normalized with respect to the corresponding CPT buckling load P_{cr}^{CPT}). The values of P_{cr} obtained through HSDT are lower than those obtained by CPT and they yield a softer response. This is reasonable since CPT neglects the transverse shear effects and can be derived from the shear deformation theories by assuming the shear and Young's moduli (i.e. G_{13}, G_{23}, E_3) to be infinite.

A further comparison between the various plate theories applied to a perfectly elastic cross-ply laminated plate (square, simply-supported) subjected to uniaxial compression is given in Fig. 2. The postbuckling results obtained from CPT, first-order shear deformation theory—FSDT ($k^2 = 5/6$) and HSDT are shown. It should be noted that in some cases the value of $k^2 = 2/3$ involved in FSDT might yield better results than those with 5/6. It is seen that the two shear deformation theories are in good agreement. An advantage of the HSDT,

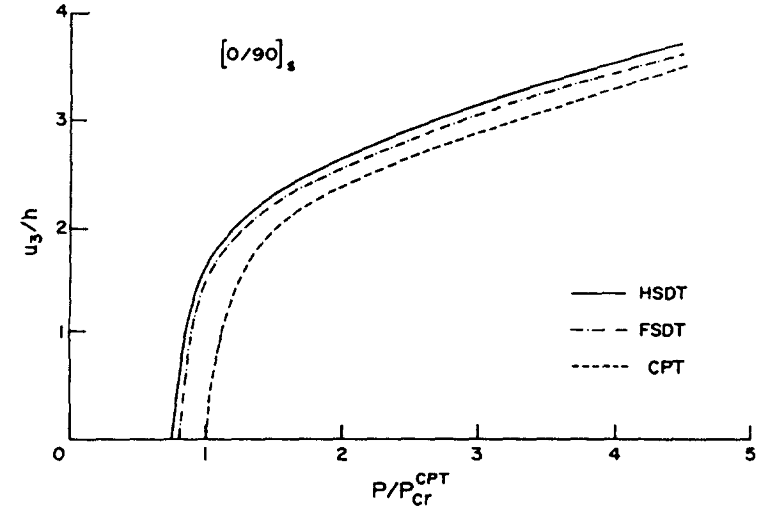

Fig. 2. Postbuckling central deflection vs load of a perfectly elastic cross-ply laminated plate (square, simply supported), generated on the basis of CPT, FSDT and HSDT.

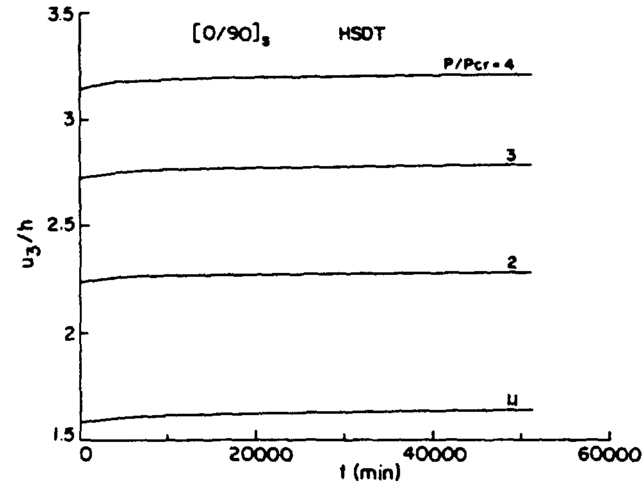

Fig. 3. Postbuckling central deflection vs time of a viscoelastic cross-ply laminated plate (square, simply supported), generated using HSDT.

however, is the lack of the need to use shear correction factors as long as the transverse shears are continuous across lamina interfaces.

The present approach is implemented to study the effect of the viscoelasticity on the postbukling behavior of cross-ply symmetric plates. The unidirectional ply consists of a boron fiber-reinforced viscoelastic epoxy matrix with a 45% fiber voflume fraction. The properties of the boron fibers are: $E^{(f)} = 443$ GPa and $\nu^{(f)} = 0.2$, which are the Young's modulus and Poisson's ratio of the material. The time-dependent Young's modulus of the viscoelastic isotropic matrix is given by (Chandiramani *et al.*, 1989):

$$E^{(m)}(t) = 0.55 + 1.24\exp(-0.4115t/1000) \quad 0 \leqslant t \leqslant 120{,}000 \text{ min},$$

where $E^{(m)}$ is measured in GPa and t in min. The Poisson's ratio was taken as 0.365.

The viscoelastic postbuckling behavior of a symmetric cross-ply plate (square, simply supported) subjected to a unidirectional compression was studied using HSDT. This behavior is displayed in Fig. 3 which exhibits the time-dependent normalized central deflection of the plate. The figure shows the response at four different load levels: $P/P_{cr} = 1.1$,

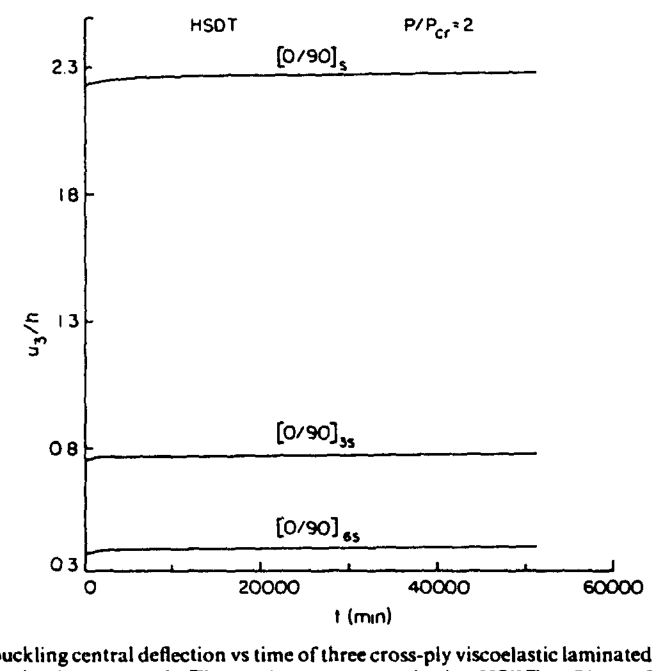

Fig. 4. Postbuckling central deflection vs time of three cross-ply viscoelastic laminated plates (square, simply supported). The results are generated using HSDT at $P/P_{cr} = 2$.

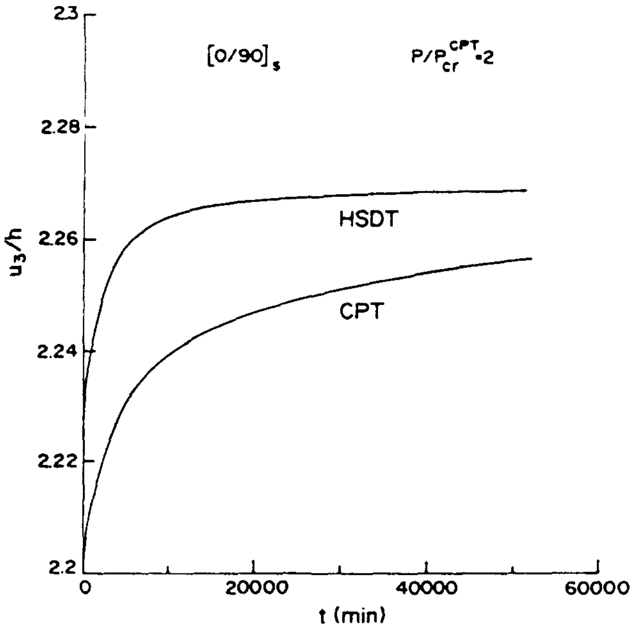

Fig. 5. Postbuckling central deflection vs time of a cross-ply viscoelastic laminated plate (square, simply supported), generated using CPT and HSDT at $P/P_{cr}^{CPT} = 2$.

2, 3, 4, where P_{cr} is the buckling load of the corresponding perfectly elastic plate (i.e. at $t = 0$ where buckling is predicted by HSDT). The effect of the viscoelastic behaviour of the plate can be well observed.

In Fig. 4, the viscoelastic effects of multi-layered laminated plates are shown for different plate thicknesses by considering $[0/90]_s$, $[0/90]_{3r}$ and $[0/90]_{6r}$ stacking sequences. The plates (square, simply supported) are subjected to uniaxial compression.

The effect of the use of CPT in analyzing a viscoelastic laminated plate is shown in Fig. 5. The plate is of the same type as used in Fig. 3. As stated before, the postbuckling behavior of the plate exhibits a softer performance when analyzed by shear deformation theories, as compared with the corresponding response obtained from a CPT theory.

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APPENDIX—THE RIGIDITY TENSORS

With $2l+1$ plies, $m+1$ as the midply and h is the ply thickness,

$$A_{\alpha 3\beta 3} = 2\left[\tilde{E}^{m+1}_{\alpha 3\beta 3} h_{m+1} + \sum_{r=1}^{l} \tilde{E}^{r}_{\alpha 3\beta 3}(h_r - h_{r+1})\right]$$

$$B_{\alpha 3\beta 3} = \frac{8}{3h^2}\left[\tilde{E}^{m+1}_{\alpha 3\beta 3} h^3_{m+1} + \sum_{r=1}^{l} \tilde{E}^{r}_{\alpha 3\beta 3}(h^3_r - h^3_{r+1})\right]$$

$$C_{\alpha\beta\xi\eta} = \frac{1}{3}\left[\tilde{E}^{m+1}_{\alpha\beta\xi\eta} h^3_{m+1} + \sum_{r=1}^{l} \tilde{E}^{r}_{\alpha\beta\xi\eta}(h^3_r - h^3_{r+1})\right]$$

$$D_{\alpha\beta\xi\eta} = \frac{4}{15h^2}\left[\tilde{E}^{m+1}_{\alpha\beta\xi\eta} h^5_{m+1} + \sum_{r=1}^{l} \tilde{E}^{r}_{\alpha\beta\xi\eta}(h^5_r - h^5_{r+1})\right]$$

$$F_{\alpha\beta\xi\eta} = \frac{2}{3}\left[\frac{\tilde{E}^{m+1}_{\alpha\beta 11}}{E^{m+1}_{1111}} E^{m+1}_{\xi 1\eta 1} h^3_{m+1} + \sum_{r=1}^{l} \frac{E^{r}_{\alpha\beta 11}}{E^{r}_{1111}} E^{r}_{\xi 1\eta 1}(h^3_r - h^3_{r+1})\right]$$

$$G_{\alpha\beta\xi\eta} = \frac{8}{15h^2}\left[\frac{E^{m+1}_{\alpha\beta 11}}{E^{m+1}_{1111}} E^{m+1}_{\xi 1\eta 1} h^5_{m+1} + \sum_{r=1}^{l} \frac{E^{r}_{\alpha\beta 11}}{E^{r}_{3111}} E^{r}_{\xi 1\eta 1}(h^5_r - h^5_{r+1})\right]$$

$$H_{\alpha\beta\xi\eta} = \tilde{E}^{m+1}_{\alpha\beta\xi\eta} h_{m+1} + \sum_{r=1}^{l} \tilde{E}^{r}_{\alpha\beta\xi\eta}(h_r - h_{r+1})$$

$$I_{\alpha\beta\xi\eta} = \frac{8}{3h^2}\left[\tilde{E}^{m+1}_{\alpha\beta\xi\eta} h^4_{m+1} + \sum_{r=1}^{l} \tilde{E}^{r}_{\alpha\beta\xi\eta}(h^4_r - h^4_{r+1})\right]$$

$$J_{\alpha\beta\xi\eta} = \frac{8}{3h^2}\left[\frac{E^{m+1}_{\alpha\beta 33}}{E^{m+1}_{3333}} E^{m+1}_{\xi 3\eta 3} h^3_{m+1} + \sum_{r=1}^{l} \frac{E^{r}_{\alpha\beta 33}}{E^{r}_{3333}} E^{r}_{\xi 3\eta 3}(h^3_r - h^3_{r+1})\right].$$